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Optimization of a two-dimensional warranty policy for a multi-component system with consideration of periodic preventive maintenance action

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Abstract

This article considers an optimal warranty policy for a multi-component system consisting of critical and non-critical components (NCCs). Each component is assumed to deteriorate as it ages and can be minimally repaired upon failure. The optimization of various types of warranty policies for repairable systems has been extensively studied and proposed by many researchers over many years. This article considers a two-dimensional warranty based on both age and usage to define the warranty period, which can be more applicable to many practical situations. In this study, we develop an optimal two-dimensional warranty policy from the manufacturer's perspective by considering not only the warranty period but also the refund/replacement (R/R) period, implementing preventive maintenance (PM) actions. The R/R of a system occurs when the number of failures of either critical components or NCCs exceeds a pre-determined threshold within a certain interval, which is usually set shorter than the warranty period. Under certain cost structures, including minimal repair cost, PM cost, and R/R cost, we develop necessary mathematical formulas to evaluate the expected total warranty costs incurred from the manufacturer's point of view and determine an optimal warranty length minimizing the expected cost rate. Numerical examples are provided to analyze the impact of several relevant factors on the optimization of the warranty policy.

Keywords: critical and non-critical component, multi-component system, optimization of warranty policy, refund/replacement occurrence, periodic preventive maintenance, age and usage, two-dimensional warranty

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1. Introduction

A warranty policy is one of the most important items to consider for both the manufacturer and the customer when the transaction of a repairable system is made. Since systems become more complex and multi-functional with multiple components and require higher reliability in performing their designated goals, not only their purchasing costs but also the costs for maintaining such systems are quite expensive. As a result, the search for the most effective maintenance and warranty strategy has been a hot issue in the field of system maintenance and warranty for a long time. Consequently, a great number of optimal maintenance strategies incorporating various types of warranty policies have been proposed and discussed in the literature by many researchers.

Classification of warranty policies can be made in many different ways. Based on whether they are renewable or not, warranties are classified as renewing warranties or non-renewing warranties. In terms of warranty cost sharing, warranties can be either free warranties or pro rata warranties. Under a free warranty, the manufacturer is fully responsible for the warranty cost, with no cost to the user, and under a pro rata warranty, the warranty cost is charged to both the manufacturer and the user with a certain proportion. Warranties can also be categorized as either one-dimensional or two-dimensional, depending on the number of factors considered. Under a two-dimensional warranty, both age and usage are considered simultaneously. Since the deterioration of many systems, such as automobiles, printers, and excavators, are jointly influenced by not only their calendar aging but also the amount of usage, a two-dimensional warranty may be considered as more practical from the viewpoint of the manufacturer/dealer. For instance, most new vehicles are warranted based on either their age or the actual distance driven, such as 5 years or 100 000 km, whichever comes first. In addition, Park *et al* [1] consider two factors—failure time and repair time—to optimize the length of the inter-preventive maintenance (PM) interval during the post-warranty period. They assert that, with failure time and repair time, it is easier to collect failure data than age and actual usage.

This article works on developing a new optimal two-dimensional warranty policy implementing periodic PM during the warranty period for a multi-component system, which is composed of critical and non-critical components (NCCs). We also assume that the component failures occur independently. Warranty and PM are two topics that have been separately studied by many researchers, but integrated research on both of them has gained significant attention in the literature [2].

PM is a planned preventive action either to keep the likelihood of system failure low or to prolong the product's remaining lifetime, while the system is still in operation. PM actions are normally performed periodically during the warranty period, with which the virtual age [3] would be reduced. Since PM is known to be a very effective maintenance strategy to slow down the deterioration of a system, manufacturers offer a certain kind of PM service for most systems at their sale. Although additional PM costs may be incurred during the warranty period from the manufacturer's perspective, it is in general beneficial for the manufacturer if the savings of the warranty cost thanks to the implementation of PM actions exceed the extra PM cost charged to the manufacturer. The periodic change of oil and its filter offered by manufacturers at the sale of vehicles is a good example of periodic PM action in practice. Dai *et al* [4] study the opportunity of different PM strategies for second-hand products covered by a two-dimensional warranty from the perspective of both dealers and customers. Su and Wang [3] work on the optimization of imperfect PM strategies, incorporating two-dimensional basic warranty and extended warranty policies. In fact, most maintenance and warranty policies premise the implementation of periodic PM during the warranty period to optimize the warranty cost. Other than periodic PM, a few research works consider situations where the inter-PM intervals vary in such a way that the older a system is, the shorter the inter-PM intervals are between two successive PM actions. When the PM is conducted, the system is inspected overall to find out any possible cause of system disruption and to maintain it so that the system is upgraded by either reducing its failure rate or age in general. At each PM action, Canfield's [5] failure reduction model restores the system's failure rate at time t to the one at $t-\alpha$, where α is a restoration level. Later, Lim *et al* [6] suggested another type of failure reduction model.

Another concept of two-dimensional warranty policy we introduce in this work is regarding the refund/replacement (R/R) of the failed system considering the PM action. In most existing warranty policies, the failed system is either replaced by a new one or is minimally repaired upon system failure during the warranty period. In this article, we set an R/R region within the warranty region, which is based on age and usage. When the component failures exceed the pre-specified upper limit of the number of component failures within such a region, either a refund or the replacement of the failed system occurs. Such an R/R warranty framework incorporating a failure number threshold is motivated by the lemon law, which is a regulation to protect customers from defective systems with repetitive failures or longer accumulated repair times. The lemon law was mainly applied to motor vehicles initially but has expanded to many other systems as well of late. The Korean type of lemon law, in particular, regulates the R/R policy for new vehicles within 1 year or 20 000 km of purchase and categorizes two types of components, critical and non-critical, to specify their respective threshold for component failures. This article considers similar R/R conditions to that of the Korean-type of lemon law to propose an optimal warranty policy.

There exist a number of maintenance and warranty models applying lemon law conditions to study optimization problems. Park *et al* [7] discuss a two-dimensional warranty based on age and usage policy applicable to lemon law conditions without considering a PM service, and the most recent references on the lemon law can be found therein. Park and Park [8] present several optimization problems regarding warranty and

maintenance policies utilizing lemon law conditions. Other studies specifically examining lemon laws include Iskandar *et al* [9], Husniah *et al* [10], and Wang *et al* [11].

The remainder of this article is structured as follows. Section 2 briefly presents the prior research works that are closely related to the warranty policy we study in this article. In section 3, we formulate a warranty modeling framework, considering the warranty and R/R regions, periodic PM actions, and a univariate approach combining age and usage. Section 4 provides several mathematical formulas to derive the expected warranty cost under the proposed two-dimensional warranty model. The optimization of the warranty length is described in section 5. To analyze the effects of several relevant variables on the optimal warranty length, we present numerical examples in section 6. Finally, concluding remarks are given in section 7.

1.1. Assumptions

(1) The system is assumed to deteriorate with increasing age.
(2) The system is composed of multiple components, each of which is categorized as either a critical component (CC) or NCC, and the system is warranted during the warranty period. Another R/R period is pre-specified within the warranty period.
(3) The CC and NCC failures occur independently.
(4) If the number of CC failures reaches l_c or the number of NCC failures reaches l_n (whichever comes first) within the R/R period, the system is either refunded or replaced to terminate the warranty. Such a condition is referred to as an R/R condition throughout this article. Each component failure is assumed to be detected instantly and is minimally repaired free to the user within the warranty period unless the R/R occurs.
(5) The occurrence of component failures follows a non-homogeneous Poisson process (NHPP).
(6) During the R/R period, a fixed number of periodic PM actions are performed at the expense of the manufacturer.
(7) The warranty expires either at the end of the warranty or at the time when the R/R occurs due to the satisfaction of R/R conditions during the R/R period. When replacement occurs, the warranty is renewed for the new system.

1.2. Nomenclature

- R/R: refund/replacement
- CC (NCC): critical (non-critical) component
- $W(U)$: age (usage) of the system
- $W_L(U_L)$: upper limit for the warranty period in terms of age (usage)
- $W_0(U_0)$: upper limit for the R/R period in terms of age (usage) with $W_0 \leqslant W_L$ and $U_0 \leqslant U_L$
- W_{RR}: elapsed time to the occurrence of R/R of the system
- $P_L(Q_L)$: age-based (usage-based) inter-PM interval
- c_{rf} : amount of refund when a refund occurs
- c_{rp}: replacement cost
- c_{pm} : unit PM cost
- $c_c(c_n)$: unit minimal repair cost for the CC (NCC)
- $U(t)$: total amount of usage of the component during $[0, t]$
- $L_c(L_n)$: number of CC (NCC) failures during the R/R period
- $l_c(l_n)$: CC (NCC) failure number threshold for occurrence of the R/R condition
- $R_u = \frac{U(t)}{t}$: random variable representing the usage rate
- $R(r) = P(R_u \leqslant r)$: cumulative distribution function of R_u
- $\lambda_i(t|r)$: conditional intensity function of the CC (NCC) for the given $R_u = r$, $i = 1, 2$
- $N_i(t|r)$: number of CC (NCC) failure during $[0, t]$ for the given $R_u = r$, $i = 1, 2$

2. Related prior works

For a two-dimensional warranty policy, the warranty period is determined by two factors, age and usage, while for a one-dimensional warranty, the length is based on only a single factor, such as the failure time of the system, without consideration of the usage. It is in general true that a two-dimensional warranty is more realistic for real-world situations since the deterioration of the system can be caused by not only by its calendar age but also its usage. Consequently, many authors have worked for several decades on the optimization of a two-dimensional warranty policy incorporating PM actions during the warranty period. Huang and Yen [12] considered both time and usage for deteriorating products in a warranty model with periodic PM actions and studied the optimal warranty policy maximizing a manufacturer's profit. Peng *et al* [13] developed an optimal PM policy by utilizing a sequence of usage-dependent failure rate thresholds under a two-dimensional warranty. Furthermore, Wang and Su [14] and Wang *et al* [15] also considered a two-dimensional warranty and developed an optimal PM strategy minimizing the total expected warranty servicing cost from a manufacturer's perspective. For additional references, Wang and Xie [16] give an excellent review of such two-dimensional warranties. Peng *et al* [17] deal with a two-dimensional PM policy with the gamma usage process, and Huang *et al* [18] study the combination of burn-in and PM for a two-dimensional warranty to determine the optimal burn-in time and amount and the optimal PM interval to minimize company loss. Salmasnia *et al* [19] and Qiao *et al* [20] also discuss a two-dimensional warranty policy incorporating PM actions.

Su and Wang [3] propose an optimal PM strategy under a two-dimensional warranty during basic and extended warranties by adopting a univariate approach. Using a univariate approach, the failure process is modeled at the system level, and the failure model defining the two-dimensional failure process can be transformed into a one-dimensional model process by introducing a usage rate, which is defined as $R_u = U(t)/t$,

where $U(t)$ represents the total usage of the system during the time interval $[0,t]$.

In this study, we extend the results of Park *et al* [7] to the situation where PM actions are implemented periodically during the R/R period in case the component failures occur independently. Park *et al* [7] propose an optimal two-dimensional warranty policy under consideration of the occurrence of either R/R subject to certain conditions with respect to the number of component failures of a multi-component system, which consists of CCs and NCCs, under the assumption that the component failures occur only internally with mutual independence. However, the PM actions are not implemented in Park *et al* [7], although the PM action is usually served by manufacturers during the warranty period as a precautionary measure. This work proposes a useful two-dimensional warranty, extending the applicability of the optimization results of Park *et al* [7].

3. Two-dimensional warranty with periodic PM

In modeling a two-dimensional warranty policy based on age and usage incorporating periodic PM actions, we adopt a virtual age modeling framework to measure the degree of the age reduction effect when the PM action is performed during the R/R period. The virtual age model was initially proposed by Kijima [21], and the discrete PM model is by Kim *et al* [22]. The work is as follows. Let $\tau_1, \tau_2, \cdots, \tau_j, \cdots$ denote the actual calendar age of the system at which the periodic PM actions are performed with $\tau_0 = 0$, where $\tau_j - \tau_{j-1} = \tau_d, j = 1,2,\cdots$, and τ_d denotes a fixed length of an inter-PM interval. At each PM action, the actual age of the system is somewhat reduced by the PM effect in terms of intensity rate, and its corresponding reduced age is referred to as a virtual age. Figure 1 describes the reduction of the virtual age at each PM action.

Let ρ_j denote the virtual age right after the jth PM action is completed for $j = 1,2,\cdots$. Then, the relation between ρ_{j-1} and ρ_j is given as [22]

$$\rho_j = \rho_{j-1} + \phi(\delta)(\tau_j - \tau_{j-1}), \tag{1}$$

where δ and $\phi(\delta)$, $\delta = 0,1,\cdots,K$, denote the level of the PM effect and the corresponding age reduction ingredient of the PM, respectively. It is noted from equation (1) that the jth PM action restores only the degradation damage experienced from τ_{j-1} to τ_j to some extent. Here, $\phi(\delta)$ is taken as a decreasing function of δ with $0 \leqslant \phi(\delta) \leqslant 1$. Throughout this article, we use $\phi(\delta) = (1+\delta)\mathrm{e}^{-\delta}$, as in Kim *et al* [22]. In the case of $\delta = 0$, $\phi(0) = 1$ and $\rho_j = \rho_{j-1} + \tau_d$, which indicates no PM effect. If $\delta = K$, the PM effect is maximized with $\phi(K) \to 0$ by letting K increase to infinity. In this case, $\rho_j = \rho_{j-1}, j \geq 1$, which implies that the jth PM action restores the system component back to as good as the previous one right after the $(j-1)^{\text{st}}$ PM action. Repeated application of equation (1) produces

$$\rho_j = \rho_{j-1} + \phi(\delta)(\tau_j - \tau_{j-1}) = \cdots = j\phi(\delta)\tau_d, \tag{2}$$

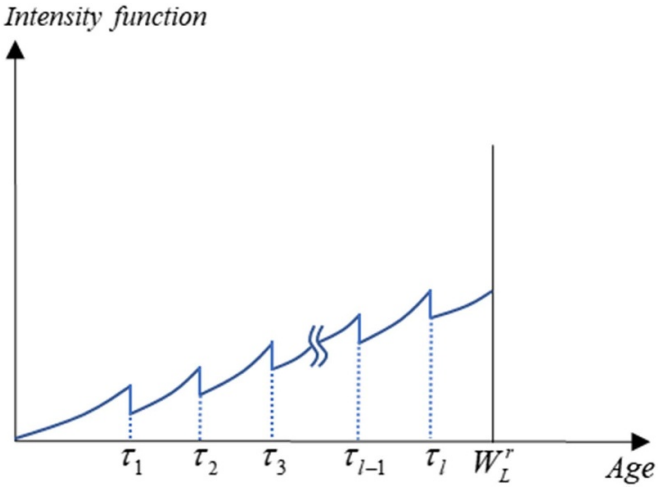


Figure 1. PM effect reducing the virtual age at each PM action.

where $\tau_d = \tau_j - \tau_{j-1}$ denotes the inter-PM interval.

This equation is used to evaluate the expected number of component failures in section 5.

3.1. Univariate approach for a two-dimensional warranty model

Several methods have been developed to formulate a failure process based on age and usage in the literature, and one of them is known as a univariate approach, which was first introduced by Murthy and Wilson [23] and later adopted in Su and Wang [3]. For a more detailed discussion on such methods, refer to Su and Wang [3] and Jack *et al* [24]. In this article, we utilize a univariate approach to model the two-dimensional failure process in terms of age and usage in a one-dimensional model by introducing a random variable representing the usage rate. Let $U\,k(t)$ denote the total usage of the component at the elapsed time kt kj and let us define the usage rate $R_u = U(t)/t$ with the cumulative distribution function of $R\ (r) = P(R_{\mathrm{u}} \leqslant \mathrm{r})$. Then, the failure process of each component can be described by defining a conditional intensity function for a given usage rate, which is dependent on both age and usage. The multi-component system we consider in this study consists of CCs and NCCs, which deteriorate as it ages. Since all component failures are assumed to be minimally repaired with negligible repair time, the CC and NCC failures occur according to the NHPP with conditional intensity functions $\lambda_i(t|r)$ for the given $R_{\mathrm{u}} = r$ for $i = 1,2$. When the usage rate is fixed, the conditional intensity function of each component may be expressed as a non-decreasing function dependent on both age and usage, and thus we set $\lambda_i(t|r) = \Lambda(t, U(t))$, where $\Lambda(\cdot,\cdot)$ is a non-decreasing function of both t and $U\ (t)$. The specific functional expressions for $\Lambda(\cdot,\cdot)$ we consider in this article are as follows:

$$\lambda_1(t|r) = \pi_1 + \pi_2 r + \pi_3 t + \pi_4 U(t) \text{ and } \lambda_2(t|r) = \pi_5 + \pi_6 r + \pi_7 t + \pi_8 U(t). \tag{3}$$

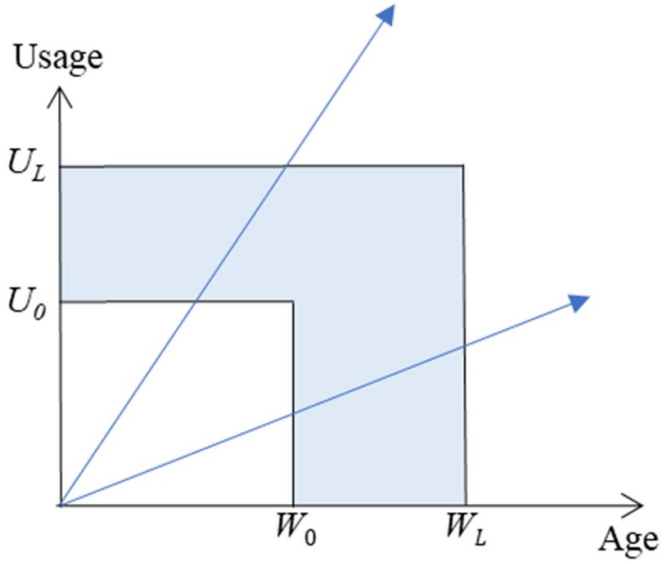


Figure 2. Warranty and R/R region under the warranty model.

Here, $U(t) = rt$ and $\pi_i's, i = 1, \cdots, 8,$ are positive constants. Su and Wang [3] and Murthy *et al* [25] have used these forms of polynomial intensity functions for numerical calculations in their work on two-dimensional warranty policies.

In the next subsection, we define an age-based R/R period and an age-based warranty period by applying a univariate approach based on the usage rate for modeling the two-dimensional warranty.

3.2. Age-based R/R period and warranty period

The two-dimensional warranty model we consider in this article is based on age and usage, each of which has an upper limit for the R/R period and warranty period. Let W_L and U_L denote the upper limits for the warranty period, and let W_0 and U_0 be the upper limits for the R/R period. Figure 2 describes the warranty and R/R regions graphically. The region defined by $[0, W_0] \times [0, U_0]$ is an R/R region, on which either a refund or replacement occurs, and the region defined by $[0, W_L] \times [0, U_L]$ is a warranty region. Under the two-dimensional warranty model, the system is either refunded or replaced to end the life cycle of the system when the number of component failures satisfies certain conditions (Assumption #4) and all component failures are minimally repaired at the expense of the manufacturer unless the R/R conditions occur within the R/R region.

Throughout this article, the usage rate of the system equals r, which remains unchanged during the life cycle of the system. As in Park *et al* [7], we adopt a univariate approach to model the failure process by defining an age-based upper limit for the warranty period and the R/R period under a two-dimensional warranty as follows. Given that $R_u = r$, the age-based upper limit for the warranty period is defined as

$$W_L^r = \min\{W_L, U_L/r\} = \begin{cases} W_L, & \text{if } r \leqslant U_L/W_L \\ U_L/r, & \text{if } r > U_L/W_L \end{cases}. \quad (4)$$

Similarly, the age-based upper limit for the R/R period can be defined as

$$W_0^r = \min\{W_0, U_0/r\} = \begin{cases} W_0, & \text{if } r \leqslant U_0/W_0 \\ U_0/r, & \text{if } r > U_0/W_0, \end{cases} \quad (5)$$

where W_L, U_L, W_0, and U_0 are the age and usage limits, as shown in figure 2.

3.3. Model formulation with consideration of PM

This work considers a repairable multi-component system that is warranted with a two-dimensional warranty based on age and usage with periodic PM actions. To reduce the two-dimensional warranty model to a one-dimensional model, we adopt a univariate approach to specify an age-based warranty period of $[0, W_L^r]$ and an age-based R/R period of $[0, W_0^r]$, as defined in equations (4) and (5) under the condition that the usage rate equals r. The warranty model works as follows. All component failures are minimally repaired free to the user during the warranty period. If the R/R condition (Assumption #4) with regard to the number of component failures occurs during the age-based R/R period, either a refund to the user is made to end the warranty or the system is replaced by a new one and the warranty is renewed. In both situations, the life cycle of the failed system is terminated. The R/R period is in general set shorter than the warranty period, and if the R/R condition does not occur during the R/R period, the system is warranted with minimal repairs on $[W_0^r, W_L^r]$ until the original warranty period is terminated. Following the expiration of the warranty, the system is maintained solely by the user.

Implementing the periodic PM actions during a two-dimensional warranty period, Wang and Su [14] and Dai *et al* [4] proposed and adopted a two-dimensional PM strategy, for which an age-based inter-PM interval is defined. Let P_a and P_u denote the inter-PM intervals based on age and usage, respectively. Consider a situation where the PM actions are

served every 6 months or every 10 000 miles by the manufacturer for a new motor vehicle. In this instance, $P_{\mathrm{a}} = 6$ months and $P_{\mathrm{u}} = 10\,000$ miles, and P_{a} and P_{u} may have different units of measurement. Applying a univariate approach similar to the warranty and R/R period, we define the age-based inter-PM interval during the age-based R/R period as

$$P_a^r = \min\{P_a, P_u/r\} = \begin{cases} P_a, & \text{if } r \leqslant P_u/P_a \\ P_u/r, & \text{if } r > P_u/P_a \end{cases} \tag{6}$$

for the given $R_{\mathrm{u}} = r$. Then, the number of periodic PM actions served over the age-based R/R period of $[0, W_0^r]$, denoted by n_{p}^r, can be expressed as

$$n_{\mathrm{p}}^r = \max\left\{j \,\middle|\, j \cdot P_a^r < W_0^r, j \geq 0\right\}. \tag{7}$$

Applying the univariate approach, the optimization problem we study in this article is reduced from an optimal two-dimensional warranty strategy to a one-dimensional warranty, as follows. Given that the usage rate equals r, the optimization problem is to find an optimal length of the original warranty period, minimizing the expected warranty cost incurred during the age-based warranty period when the periodic PM actions are conducted n_{p}^r times during the age-based R/R period. No PM actions are performed on $[W_0^r, W_L^r]$ following the R/R period, and only minimal repairs by the manufacturer are served upon the component failures.

4. Expected warranty cost under a two-dimensional warranty

To evaluate the expected warranty cost during the age-based warranty period, we first need to find the expected number of CC and NCC failures, given that the usage rate is fixed at r. Let l_{c} and l_{n} denote the failure number thresholds for CCs and NCCs for the occurrence of the R/R condition, respectively. Then, either a refund or replacement of the system occurs if the number of CC failures reaches l_{c} or the number of NCC failures reaches l_{n}, whichever comes first, during the R/R period. Otherwise, no refund or replacement occurs until the warranty expires.

Let $N_1^r(t)\,(N_2^r(t))$ denote the random variables representing the number of CC (NCC) failures during the interval $[0,t]$ within the age-based R/R period for the given $R_{\mathrm{u}} = r$, and the periodic PM actions are conducted n_{p}^r times with the inter-PM interval of P_{a}^r until the R/R period ends. Since the CC (NCC) failures follow an NHPP with an intensity function $\lambda_1(t|r)$ $[\lambda_2(t|r)]$, the expected number of CC failures in the interval $[0.W_0^r]$ can be obtained by applying equation (2), as follows:

$$\mathrm{E}[N_1^r(W_0^r)] = \sum_{j=0}^{n_{\mathrm{p}}^r - 1} \int_{j\phi(\delta)\tau_d}^{j\phi(\delta)\tau_d + \tau_d} \lambda_1(t|r)\,\mathrm{d}t + \int_{n_{\mathrm{p}}^r\phi(\delta)\tau_d}^{n_{\mathrm{p}}^r\phi(\delta)\tau_d + W_0^r - n_{\mathrm{p}}^r\tau_d} \lambda_1(t|r)\,\mathrm{d}t. \tag{8}$$

Here, the second term indicates the expected number of failures occurring between the n_{p}^rth PM time and the end of the warranty period. Similarly, the expected number of NCC failures is expressed as

$$\mathrm{E}[N_2^r(W_0^r)] = \sum_{j=0}^{n_{\mathrm{p}}^r - 1} \int_{j\phi(\delta)\tau_d}^{j\phi(\delta)\tau_d + \tau_d} \lambda_2(t|r)\,\mathrm{d}t + \int_{n_{\mathrm{p}}^r\phi(\delta)\tau_d}^{n_{\mathrm{p}}^r\phi(\delta)\tau_d + W_0^r - n_{\mathrm{p}}^r\tau_d} \lambda_2(t|r)\,\mathrm{d}t. \tag{9}$$

By utilizing equations (8) and (9), the probability distribution for $N_1^r(W_0^r)\,[N_2^r(W_0^r)]$ can be obtained as

$$P[N_1^r(W_0^r) = l] = \frac{\{E[N_1^r(W_0^r)]\}^l}{l!} \cdot \mathrm{e}^{-\{E[N_1^r(W_0^r)]\}} \tag{10}$$

and

$$P[N_2^r(W_0^r) = l] = \frac{\{E[N_2^r(W_0^r)]\}^l}{l!} \cdot \mathrm{e}^{-\{E[N_2^r(W_0^r)]\}}$$

for $l = 1, 2, \cdots$.

The R/R condition for the occurrence of either a refund or replacement is satisfied if there exists a time point $t \in [0, W_0^r]$ at which one of the following two conditions holds:

$$\begin{aligned} &\text{(i)}\, N_1^r(t) = l_{\mathrm{c}} \text{ and } N_2^r(\mathrm{t}) \leqslant l_{\mathrm{n}} - 1,\ t \in [0, W_0^r] \\ &\text{(ii)}\, N_2^r(t) = l_{\mathrm{n}} \text{ and } N_1^r(t) \leqslant l_{\mathrm{c}} - 1,\ t \in [0, W_0^r] \end{aligned} \tag{11}$$

during the age-based R/R period of $[0, W_0^r]$. In the next subsections, we derive the expected warranty costs for both refund and replacement separately.

4.1. Refund warranty scheme

When a refund to the user is made subject to the satisfaction of the R/R condition, the amount of the refund is either the purchasing price of the new system or is pro rata in proportion to the usage. Then, the life cycle of the system ends when the refund is made and the warranty expires. Let W_{RR} denote the elapsed time for the occurrence of the R/R condition within the age-based R/R period. The warranty costs during the age-based warranty period can be evaluated in three different situations: condition (i) and condition (ii) of equation (11), and no occurrence of the R/R condition. For notational convenience, we define the events $A = \{\text{condition (i) occurs}\}$, $B = \{\text{condition (ii) occurs}\}$, and $C = \{\text{no occurrence of the R/R condition}\}$. Let $C_{\mathrm{c}}\,(C_{\mathrm{n}})$ denote the random variable representing the warranty cost when the R/R condition occurs due to CC (NCC) failures, and let C_{nr} be the warranty cost when no R/R condition occurs. Then, for the given $R_{\mathrm{u}} = r$, the total warranty cost incurred until the expiration of the warranty, denoted by $C_1(W_0^r, W_L^r)$, can be evaluated as

$$C_1(W_0^r, W_L^r) = C_{\mathrm{c}} \cdot I(A) + C_{\mathrm{n}} \cdot I(B) + C_{\mathrm{nr}} \cdot I(C), \tag{12}$$

where $I(\cdot)$ is an indicator function defined as $I(E)=1$ if an event E occurs, $I(E)=0$ otherwise, and

$$\begin{aligned} C_{\mathrm{c}} &= c_{\mathrm{c}}\cdot(l_{\mathrm{c}}-1)+c_{\mathrm{rf}}+c_{\mathrm{pm}}\cdot n_{\mathrm{p}}^{r} \\ &\quad + c_{\mathrm{n}}\cdot\{N_2^r(W_{\mathrm{RR}})\,|N_2^r(W_{\mathrm{RR}})\leqslant l_{\mathrm{n}}-1\}, \\ C_{\mathrm{n}} &= c_{\mathrm{n}}\cdot(l_{\mathrm{n}}-1)+c_{\mathrm{rf}}+c_{\mathrm{pm}}\cdot n_{\mathrm{p}}^{r} \\ &\quad + c_{\mathrm{c}}\cdot\{N_1^r(W_{\mathrm{RR}})\,|N_1^r(W_{\mathrm{RR}})\leqslant l_{\mathrm{c}}-1\}, \end{aligned}$$

and

$$\begin{aligned} C_{\mathrm{nr}} = {} & [c_{\mathrm{c}}\cdot\{N_1^r(W_0^r)\,|N_1^r(W_0^r) \\ & \leqslant l_{\mathrm{c}}-1\}+c_{\mathrm{n}}\cdot\{N_2^r(W_0^r)\,|N_2^r(W_0^r) \\ & \leqslant l_{\mathrm{n}}-1+c_{\mathrm{c}}\cdot\{\text{no. of CC failures on } [W_0^r,W_L^r]\} \\ & + c_{\mathrm{n}}\cdot\{\text{no. of NCC failures on } [W_0^r,W_L^r]\}+c_{\mathrm{pm}}\cdot n_{\mathrm{p}}^{r}. \end{aligned}$$

Here, $c_{\mathrm{c}}\,(c_{\mathrm{n}})$ denotes the unit minimal repair cost for CC (NCC) failures, c_{pm} is the unit PM cost, and c_{rf} is the refund cost when it occurs. By taking the expectation on $C_1(W_0^r,W_L^r)$ of equation (12), we obtain $E\{C_1(W_0^r,W_L^r)\}$, the expected total warranty cost. However, the distribution of W_{RR} has not been solved, and thus $E\{N_1^r(W_{\mathrm{RR}})\,|N_1^r(W_{\mathrm{RR}})\leqslant l_{\mathrm{c}}-1\}$ and $E\{N_2^r(W_{\mathrm{RR}})\,|N_2^r(W_{\mathrm{RR}})\leqslant l_{\mathrm{n}}-1\}$ may not be obtainable (see Remark 1 below). Thus, we replace W_{RR} with W_0^r and use the relation

$$E\{N_1^r(W_0^r)\,|N_1^r(W_0^r)\leqslant l_{\mathrm{c}}-1\}=\sum_{i=1}^{l_{\mathrm{c}}-1} iP\{N_1^r(W_0^r)=i\}$$

and

$$E\{N_2^r(W_0^r)\,|N_2^r(W_0^r)\leqslant l_{\mathrm{n}}-1\}=\sum_{j=1}^{l_{\mathrm{n}}-1} jP\{N_2^r(W_0^r)=j\} \quad (13)$$

to obtain $EC_{\mathrm{c}}, EC_{\mathrm{n}}$, and EC_{nr} by utilizing equation (10).

Remark 1. In this article, we define a fixed length of an age-based R/R period, denoted by W_0^r, and if the R/R conditions occur at elapsed time $W_{\mathrm{RR}}\,(\leqslant W_0^r)$ within the R/R period, the system is either refunded or replaced to end the warranty. Otherwise, the warranty expires at the end of the age-based warranty period. Thus, we need to derive the probability distribution of W_{RR} to obtain $N_1^r(E\{W_{\mathrm{RR}}\})$ or $N_2^r(E\{W_{\mathrm{RR}}\})$ for EC_{c}, EC_{n}, and EC_{nr} in the following. However, the exact distribution of W_{RR} cannot be derived analytically at the present time, and thus we substitute W_0^r for W_{RR} in equations (13) and (14). For the same reason, $c_{\mathrm{pm}}\cdot n_{\mathrm{p}}^r$ is used to evaluate the PM costs in equations (14) and (17). Note that n_{p}^r is the number of periodic PM actions performed on $[0,W_0^r]$, not on $[0,E\{W_{\mathrm{RR}}\}]$. In addition, W_0^r is used instead of $E\{W_{\mathrm{RR}}\}$ in the formula for $E\{L_W^r\}$ in section 5 as well.

For better approximation for $E\{W_{\mathrm{RR}}\}$, we may use $E\{W_{\mathrm{nRR}}\}$ instead of W_0^r, where W_{nRR} denotes the elapsed time to the refund or replacement in the case of no PM actions during the R/R period. $E\{W_{\mathrm{nRR}}\}$ can be formulated as follows. For $\mathrm{t}\in[0,\,W_0^r]$,

$$\begin{aligned} \bar{H}(t) &= P(W_{n\mathrm{RR}}>t)=P(N_{n1}^r(t)\leqslant l_{\mathrm{c}}-1)\,P(N_{n2}^r(t)\leqslant l_{\mathrm{n}}-1) \\ &= \sum_{i=1}^{l_{\mathrm{c}}-1}\frac{\left[\int_0^t\lambda_1\left(u\middle|r\right)\mathrm{d}u\right]^i \mathrm{e}^{-\int_0^t\lambda_1\left(u\middle|r\right)\mathrm{d}u}}{i!} \\ &\quad\cdot\sum_{j=1}^{l_{\mathrm{n}}-1}\frac{\left[\int_0^t\lambda_2\left(u\middle|r\right)\mathrm{d}u\right]^j \mathrm{e}^{-\int_0^t\lambda_2\left(u\middle|r\right)\mathrm{d}u}}{j!}, \end{aligned}$$

and it follows that $E\{W_{\mathrm{nRR}}\}=\int_{W_0^r}^{0}\bar{H}(t)\,\mathrm{d}t$. Here, $N_{n1}^r(t)$ and $N_{n2}^r(t)$ denote the numbers of CC and NCC failures on $[0,t]$ in the case of no PM actions. Note that $E\{W_{\mathrm{RR}}\}\leqslant E\{W_{\mathrm{nRR}}\}\leqslant W_0^r$. Thus, $E\{W_{\mathrm{nRR}}\}$ may be substituted for W_0^r in equation (13) to reduce the errors for $E\{N_1^r(W_0^r)\,|N_1^r(W_0^r)\leqslant l_{\mathrm{c}}-1\}$ and $E\{N_2^r(W_0^r)\,|N_2^r(W_0^r)\leqslant l_{\mathrm{n}}-1\}$.

By the independence of $N_1^r(W_0^r)$ and $N_2^r(W_0^r)$, we have $\mathrm{P}(N_1^r(W_0^r)=i,N_2^r(W_0^r)=j) = P(N_1^r(W_0^r)=i)\,P(N_2^r(W_0^r)=j)$, and thus we obtain the following formulas:

$$\begin{aligned} EC_{\mathrm{c}} &= \left\{c_{\mathrm{c}}\cdot(l_{\mathrm{c}}-1)+c_{\mathrm{rf}}+c_{\mathrm{pm}}\cdot n_{\mathrm{p}}^r+c_{\mathrm{n}}\sum_{j=1}^{l_{\mathrm{n}}-1}jP\{N_2^r(W_0^r)=j\}\right\} \\ &\quad\times P(N_1^r(W_0^r)=\ l_{\mathrm{c}})\,P(N_2^r(W_0^r)\leqslant l_{\mathrm{n}}-1), \\ EC_{\mathrm{n}} &= \left\{c_{\mathrm{n}}\cdot(l_{\mathrm{n}}-1)+c_{\mathrm{rf}}+c_{\mathrm{pm}}\cdot n_{\mathrm{p}}^r+c_{\mathrm{c}}\sum_{j=1}^{l_{\mathrm{c}}-1}iP\{N_1^r(W_0^r)=i\}\right\} \\ &\quad\times P(N_1^r(W_0^r)\leqslant l_{\mathrm{c}}-1)\,P(N_2^r(W_0^r)=l_{\mathrm{n}}), \end{aligned}$$

and

$$\begin{aligned} EC_{\mathrm{nr}} = \Bigg\{ & c_{\mathrm{c}}\left[\sum_{j=1}^{l_{\mathrm{c}}-1}iP\{N_1^r(W_0^r)=i\}+\int_{W_0^r}^{W_L^r}\lambda_1\left(t\middle|r\right)\mathrm{d}t\right] \\ & +c_{\mathrm{n}}\left[\sum_{j=1}^{l_{\mathrm{n}}-1}jP\{N_1^r(W_0^r)=\mathrm{j}\}+\int_{W_0^r}^{W_L^r}\lambda_2\left(t\middle|r\right)\mathrm{d}t\right]+c_{\mathrm{pm}}\cdot n_{\mathrm{p}}^r\Bigg\} \\ & \cdot P(N_1^r(W_0^r)\leqslant l_{\mathrm{c}}-1)\,P(N_2^r(W_0^r)\leqslant l_{\mathrm{n}}-1). \end{aligned} \quad (14)$$

Thus, by summing up $EC_{\mathrm{c}}, EC_{\mathrm{n}}$, and EC_{nr}, we obtain a formula to evaluate the expected total warranty cost incurred during the age-based warranty period as

$$E\{C_1(W_0^r,W_L^r)\}=\ \mathrm{EC}_{\mathrm{c}}+\mathrm{EC}_{\mathrm{n}}+EC_{\mathrm{nr}}. \quad (15)$$

Suppose that the usage rate R_{u} is a random variable with the cumulative distribution function of $R(\cdot)$ and the support of $[r_{\min},r_{\max}]$. Then, by taking the integration of $E\{C_1(W_0^r,W_L^r)\}$ with respect to r, we obtain the expected total warranty cost as

$$E\{C_1(W_0,W_L)\}=\int_{r_{\max}}^{r_{\min}}E\{C_1(W_0^r,W_L^r)\}\,\mathrm{d}R(r). \quad (16)$$

4.2. Replacement warranty scheme

When replacement occurs at W_{RR} due to the satisfaction of the R/R condition during the age-based R/R period, the failed system is replaced by a new system at the expense of the manufacturer, and the warranty is renewed with exactly the same contract terms as the original one. In the case of replacement, the expected total warranty cost, denoted by $E\{C_2(W_0^r, W_L^r)\}$, can be evaluated similarly to that of equation (15). For simplification of the equations, we denote $E_R = E\{C_2(W_0^r, W_L^r)\}$. By adding an extra term $E_R = E\{C_2(W_0^r, W_L^r)\}$ in the formulas for EC_{c} and EC_{n}, given in equation (14), we have

$$EC_{\mathrm{c}}^* = \left\{ c_{\mathrm{c}} \cdot (l_{\mathrm{c}} - 1) + c_{rp} + c_{\mathrm{pm}} \cdot n_{\mathrm{p}}^r + c_{\mathrm{n}} \sum_{j=1}^{l_{\mathrm{n}}-1} jP\{N_2^r(W_0^r) = j\} + E_R \right\} \times P(N_1^r(W_0^r) = l_{\mathrm{c}}) \cdot P(N_2^r(W_0^r) \leqslant l_{\mathrm{n}} - 1),$$

$$EC_{\mathrm{n}}^* = \left\{ c_{\mathrm{n}} \cdot (l_{\mathrm{n}} - 1) + c_{rp} + c_{\mathrm{pm}} \cdot n_{\mathrm{p}}^r + c_{\mathrm{c}} \sum_{i=1}^{l_{\mathrm{c}}-1} iP\{N_1^r(W_0^r) = i\} + E_R \right\} P(N_1^r(W_0^r) \leqslant l_{\mathrm{c}} - 1) \cdot P(N_2^r(W_0^r) = l_{\mathrm{n}}). \tag{17}$$

Here, EC_{c}^* and EC_{n}^* are the expected warranty costs when the R/R condition occurs due to CC failures and NCC failures, respectively. To reflect the renewal of the warranty for the replaced system, the term $E_R = E\{C_2(W_0^r, W_L^r)\}$ is added in equation (17). Thus, we have

$$E_R = EC_{\mathrm{c}}^* + EC_{\mathrm{n}}^* + EC_{\mathrm{nr}}, \tag{18}$$

where EC_{nr} is given in equation (14). Solving equation (18) for $E_R = E\{C_2(W_0^r, W_L^r)\}$, we obtain

$$\mathrm{E}\{C_2(W_0^r, W_L^r)\} = \frac{\mathrm{E}\{C_1(W_0^r, W_L^r)\}}{1 - P(N_1^r(W_0^r) = l_{\mathrm{c}})P(N_2^r(W_0^r) \leqslant l_{\mathrm{n}} - 1) - P(N_1^r(W_0^r) \leqslant l_{\mathrm{c}} - 1)P(N_2^r(W_0^r) = l_{\mathrm{n}})}. \tag{19}$$

As in equation (16) for the refund situation, the expected total warranty cost over the warranty period can be obtained as

$$E\{C_2(W_0, W_L)\} = \int_{r_{\min}}^{r_{\max}} E\{C_2(W_0^r, W_L^r)\}\,\mathrm{d}R(r). \tag{20}$$

5. Optimization of the warranty length

The warranty expires either at W_{RR}, the elapsed time to the occurrence of the R/R conditions within the R/R period, or at W_L^r, the end of the age-based warranty period. Let L_{W} denote the length of the warranty period under the two-dimensional warranty policy we study in this article. Then, for the given $R_{\mathrm{u}} = r$, L_{W}^r is defined as

$$L_W^r = \begin{cases} W_{\mathrm{RR}}, & \text{if the R/R condition occurs on } [0, W_0^r] \\ W_L^r, & \text{if no R/R condition occurs} \end{cases}.$$

Thus, we have

$$\mathrm{E}\{L_W^r\} = E(W_{\mathrm{RR}})P(\mathrm{R/R\,condition}) + W_L^r P(\mathrm{no\,R/R\,condition}).$$

However, since $E(W_{\mathrm{RR}})$ has not been obtained in this study when implementing periodic PM actions, we replace $E(W_{\mathrm{RR}})$ with W_0^r to evaluate the expected warranty length. For further discussions of substituting W_0^r for $E(W_{\mathrm{RR}})$, see Remark 1. Applying the fact that

$$P(\mathrm{no\,R/R\,condition}) = P(N_1^r(W_0^r) \leqslant l_{\mathrm{c}} - 1) \times P(N_2^r(W_0^r) \leqslant l_{\mathrm{n}} - 1),$$

the expected length of the warranty can be evaluated as

$$\mathrm{E}\{L_W^r\} = W_0^r \cdot P(\mathrm{R/R\,condition}) + W_L^r \cdot P(\mathrm{no\,R/R\,condition}),$$

where $P(R/R\,\mathrm{condition}) = 1 - P(\mathrm{no\,R/R\,condition})$. By taking the integration of $E\{L_{\mathrm{W}}^r\}$ with respect to r, we have

$$E(L_{\mathrm{W}}) = \int_{r_{\max}}^{r_{\min}} E\{L_{\mathrm{W}}^r\}\,\mathrm{d}R(r). \tag{21}$$

By using equations (16)–(21), the expected cost rate (ECR) for both refund and replacement schemes, respectively, is given by,

$$\mathrm{ECR}[C_i(W_0, W_L)] = \frac{\mathrm{E}[C_i(W_0, W_L)]}{\mathrm{E}[L_W]}, \tag{22}$$

where $i = 1$ for a refund and $i = 2$ for a replacement.

Hence, the optimization problem is to determine the optimal warranty length of the original warranty policy so as to minimize $\mathrm{ECR}[C_i(W_0, W_L)]$, given in equation (22). The optimal warranty length, denoted by W_L^*, satisfies

$$\mathrm{ECR}[C_i(W_0, W_L^*)] = \min_{W_L} \mathrm{ECR}[C_i(W_0, W_L)] \tag{23}$$

for $i = 1$ and 2. If the value of r is known to be equal to r_0, that is $r = r_0$, then $W_L^*(r_0)$ is to be determined by

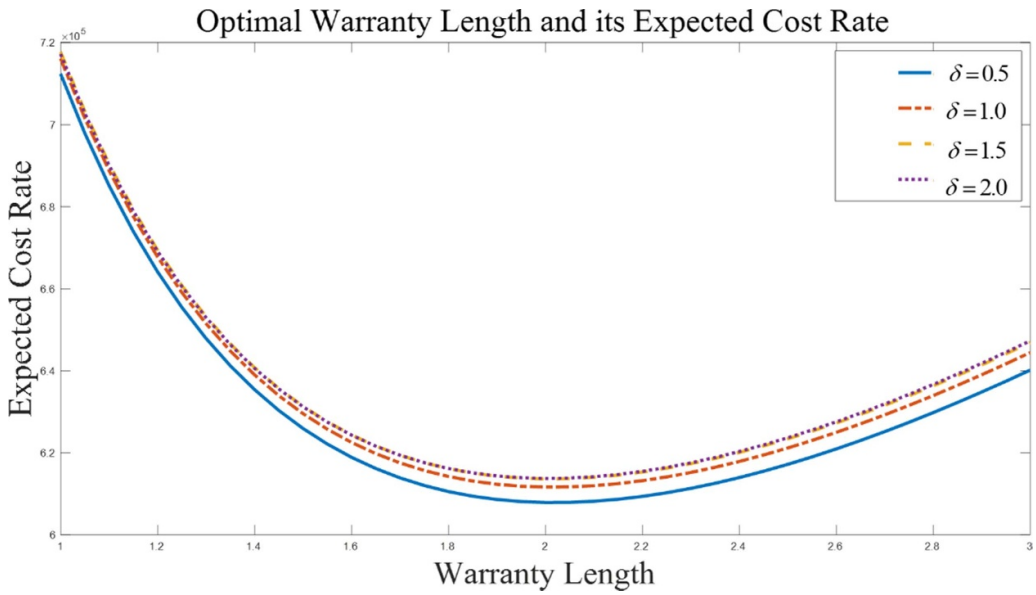


Figure 3. Optimal warranty length and its corresponding ECR for various values of δ with $c_{\rm c} = 2,000$, $c_{\rm n} = 400$, $c_{\rm pm} = 2000$, $a_r/\mu_T = 1/2$, $l_{\rm c} = 2$, $l_{\rm n} = 4$, and $\tau_d = 0.3$.

$$\mathrm{ECR}\left[C_i\left(W_0, W_L^*\left(r_0\right)\right)\right] = \min_{W_L} \mathrm{ECR}\left[C_i\left(W_0, W_L^{r_0}\right)\right]. \quad (24)$$

Remark 2. The optimal age limit, W_L^*, is obtained by equation (23). Another interesting question would be 'Can W_L^* be utilized to find W_0^* ?'. One possible heuristic solution is to use the relation of 'usage $= r \times$ age' to obtain $U_L^* = W_L^* \times \int_{r_{\min}}^{r_{\max}} r\mathrm{d}R\left(r\right)$. More rigorously, an objective function for the optimization problem based on the usage-based warranty period and the other R/R period is defined, and similar methods to those adopted in this article may be used to obtain an optimal usage limit.

6. Numerical example

Under the proposed two-dimensional warranty policy, the optimal warranty length is determined depending on several parameters, such as the length of the R/R period, the inter-PM interval, the repair cost, and the R/R conditions. In this section, we numerically conduct a sensitivity analysis to demonstrate the impact of several relevant parameters on the optimal value of the original warranty length, which is the critical factor for designing a two-dimensional warranty policy from a manufacturer's perspective. Since the objective functions, given in equation (22), are intractable to obtain the unique optimal solutions analytically, we utilize the Nelder–Mead downhill simplex method for numerical calculations in this section. A software package for use of the Nelder–Mead method is readily available to numerically obtain extreme values for complex non-linear objective functions. For the numerical calculations, we first designate the initial conditional intensity functions, $\lambda_1\left(t\middle|r\right)$ and $\lambda_2\left(t\middle|r\right)$ for CCs and NCCs given in equation (3), with the parameter values of $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0.7$ and $\pi_5 = \pi_6 = \pi_7 = \pi_8 = 0.8$ and set $U\left(t\right) = rt$. In addition, we set $U_L = rW_L$, $U_0 = rW_0$, and $P_a = rP_u$ in equations (4)–(6) for simplification of the numerical calculations so that the relations $W_L^r = W_L$ and $W_0^r = W_0$ hold, not dependent on the value of r, throughout this section. Thus, as an objective function for the optimization problem to determine the optimal warranty length for the numerical analysis, we use $\mathrm{ECR}\left[C_i\left(W_0, W_L\right)\right]$, given in equation (22), where $E\left[C_i\left(W_0, W_L\right)\right]$ and $E\left[L_{\rm W}\right]$ are calculated by substituting W_L, W_0, and $P_{\rm a}$ for W_L^r, W_0^r, and $P_{\rm a}^r$, respectively.

For numerical calculations for sensitivity analysis in this section, we consider only the refund situation, and all the tables and figures are made based on the objective function of $\mathrm{ECR}\left[C_1\left(W_0, W_L\right)\right]$. When the refund occurs within the R/R period, the amount of refund to the user is determined proportionately to the usage of the system by adopting the following equation:

$$c_{\rm rf} = p_s \cdot \left(1 - \frac{a_v}{\mu_T}\right) + c_0,$$

where p_s, μ_T, a_v, and c_0 denote the original purchasing price of the system, mean life length, virtual age at the occurrence of the refund, and other basic costs, respectively. We assume that $\mu_{\rm T}$ and $a_{\rm r}$ are assumed to be known, and $a_{\rm v} < \mu_{\rm T}$ since the length of the R/R period is usually set shorter than the mean life length of the system. Park *et al* [7] have also used a similar formula to evaluate the refund amount.

For the numerical calculations in this section, we take $p_{\rm s} = 30\,000$, and $c_0 = 300$, and let $\frac{a_{\rm r}}{\mu_{\rm T}} = 0.5$, which implies that the refund occurs at half of the mean life length of the multi-component system. We also set $W_0 = 12\,\mathrm{month}\left(1.0\,\mathrm{year}\right)$, which implies that the length of the R/R period equals 1.0 year, and thus either a refund or replacement occurs within a 1-year period.

Due to the complex non-linear functional expression of the objective function $\mathrm{ECR}\left[C_i\left(W_0, W_L\right)\right]$, given in equation (22), the existence and uniqueness of the solution has not been proven analytically in this study. However, figures 3 and 4

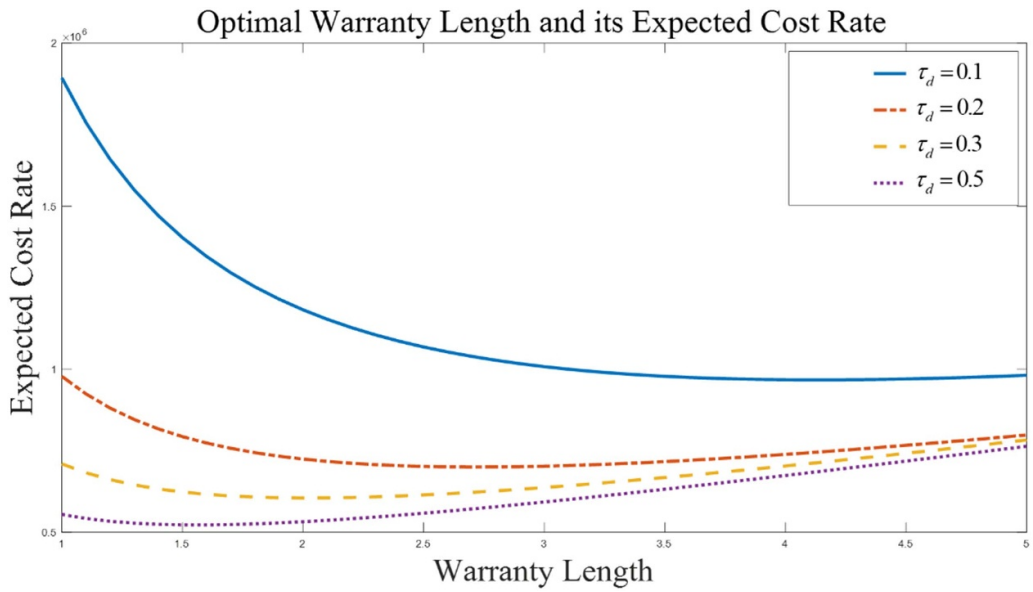


Figure 4. Optimal warranty length and its corresponding ECR for various values of τ_d with $c_c = 2,000$, $c_n = 400, c_{pm} = 2000$, $a_r/\mu_T = 1/2$, $l_c = 2, l_n = 4$, and $\delta = 0.5$.

Table 1. Optimal warranty length W_L^* and its corresponding ECR for various combinations of l_c and c_c for $c_n = 400, c_{pm} = 500$, $l_n = 4, W_0 = 1.0, \delta = 0.1,$ and $\tau_d = 0.2$.

	$c_c = 500$		$c_c = 1000$		$c_c = 1500$		$c_c = 2000$	
l_c	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR
2	2.73	176 122	1.60	244 999	0.96	276 780	0.31	287 565
3	2.85	327 453	1.83	480 932	1.33	592 578	0.98	672 677
4	2.96	418 994	2.01	636 256	1.57	814 499	1.30	969 513
5	3.01	460 988	2.10	711 162	1.69	924 596	1.44	1118 040
6	3.05	476 503	2.14	739 757	1.74	967 383	1.50	1176 260

Table 2. Optimal warranty length W_L^* and its corresponding ECR for various combinations of l_c and c_{pm} for $c_c = 1000, c_n = 400$, $l_n = 4, W_0 = 1.0, \delta = 0.1,$ and $\tau_d = 0.2$.

	$c_{pm} = 1000$		$c_{pm} = 1100$		$c_{pm} = 1200$		$c_{pm} = 1500$	
l_c	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR
2	2.72	350 848	2.89	367 135	3.06	382 502	3.50	424 341
3	2.85	653 703	3.01	681 304	3.17	707 489	3.59	779 336
4	2.96	836 928	3.12	869 764	3.26	901 039	3.67	987 337
5	3.02	920 977	3.17	955 707	3.39	989 029	3.71	1080 580
6	3.05	952 031	3.20	987 330	3.43	1021 320	3.73	1114 470

show graphical representations of the convex functional relation of the ECR versus warranty length for various values of δ and τ_d, respectively. Thus, we conjecture that there exists a unique solution, W_L^*, which minimizes the ECR under our proposed warranty model.

Tables 1 and 2 present the optimal warranty length, denoted by W_L^*, and its corresponding ECR for various combinations of l_c, c_c, and c_{pm} to study the impact of the failure number threshold, the unit minimal repair cost, and the unit PM cost for the CC failures on the warranty model. From both tables, we observe that as the failure number threshold or the unit PM cost increases, the optimal warranty length becomes steadily longer. Such trends imply that the greater the threshold is, the longer a warranty period should be in order to be more beneficial to the manufacturer with regard to the ECR during the warranty period. However, W_L^* decreases as the minimal repair cost becomes more expensive. Obviously, the ECR becomes higher as either the threshold or the unit repair and PM costs increase. It is interesting to observe from tables 3 and 4 that W_L^* becomes shorter as the failure number threshold for NCC failures increases regardless of the unit minimal repair and PM costs. It is also observed from table 4 that, as the unit PM cost increases, the optimal warranty length becomes longer. In general, it seems from tables 1–4 that a longer warranty length is beneficial to the manufacturer as the unit PM cost increases, while a shorter warranty length is preferred as the unit minimal repair cost for CCs or NCCs becomes higher. The failure number thresholds for CCs and NCCs work differently to determine the optimal warranty length.

To investigate the impact of periodic PM actions on the proposed warranty model, W_L^* and its ECR are calculated for various combinations of l_c, l_n, τ_d, and δ. Table 5 tabulates the numerical results for W_L^* and its ECR by considering the inter-PM interval, denoted by $\tau_d = \tau_j - \tau_{j-1}$. It is observed, for instance, from table 5 that, when the R/R period is set to 1.0 year $(W_0 = 1.0)$ and the PM actions are performed every 0.3 year $(\tau_d = 0.3)$ with a 0.1 level of PM effect $(\delta = 0.1)$, the optimal warranty length equals 1.49 years with its ECR of 575 742 for the CC (NCC) failure number threshold of $4\,(8)$. Table 5 shows that W_L^* increases when l_c increases and each of l_n and τ_d gets decreased. Thus, it is suggested that as the PM actions are performed more often (as τ_d decreases), a longer warranty length is more beneficial to the manufacturer. As τ_d decreases or each of l_c and l_n increases, the ECR becomes higher, which is as anticipated. It is observed from tables 5

Table 3. Optimal warranty length W_L^* and its corresponding ECR for various combinations of l_n and c_n for $c_c = 1000, c_{pm} = 500$, $l_c = 2, W_0 = 1.0, \delta = 0.1,$ and $\tau_d = 0.2$.

	$c_n = 200$		$c_n = 300$		$c_n = 500$		$c_n = 700$	
l_n	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR
2	1.72	52 563	1.69	78 005	1.66	128 871	1.65	179 731
3	1.66	97 049	1.64	144 522	1.62	239 454	1.61	334 381
4	1.62	123 655	1.61	184 330	1.59	305 668	1.59	426 998
5	1.61	136 019	1.59	202 835	1.58	336 453	1.58	470 066
6	1.60	140 765	1.59	209 938	1.57	348 270	1.57	486 595

Table 4. Optimal warranty length W_L^* and its corresponding ECR for various combinations of l_n and c_{pm} for $c_c = 1000, c_n = 400$, $l_c = 2, W_0 = 1.0, \delta = 0.1,$ and $\tau_d = 0.2$.

	$c_{pm} = 1000$		$c_{pm} = 1100$		$c_{pm} = 1200$		$c_{pm} = 1500$	
l_n	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR
2	2.79	146 829	2.96	153 557	3.13	159 912	3.58	177 238
3	2.75	274 124	2.92	286 796	3.08	298 757	3.53	331 337
4	2.72	350 848	2.89	367 135	3.06	382 502	3.50	424 341
5	2.71	386 739	2.88	404 731	3.04	421 704	3.48	467 901
6	2.70	400 590	2.87	419 244	3.04	436 841	3.48	484 729

Table 5. Optimal warranty length W_L^* and its corresponding ECR for various combinations of l_c, l_n, and τ_d for $c_c = 1000, c_n = 400$, $c_{pm} = 500, W_0 = 1.0,$ and $\delta = 0.1$.

		$\tau_d = 0.1$		$\tau_d = 0.2$		$\tau_d = 0.3$	
l_c	l_n	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR
	4	2.85	653 720	1.83	480 932	1.30	393 383
	5	2.84	722 890	1.82	531 333	1.29	424 130
3	6	2.83	749 924	1.81	550 976	1.29	434 635
	7	2.83	758 953	1.81	557 518	1.28	437 699
	8	2.83	761 593	1.81	559 427	1.28	438 482
	5	2.95	927 629	2.00	704 940	1.50	556 021
	6	2.94	963 410	1.99	732 004	1.49	570 409
4	7	2.94	975 453	1.99	741 105	1.49	574 650
	8	2.94	978 998	1.99	743 782	1.49	575 742
	9	2.94	979 927	1.99	744 482	1.49	575 992
	6	3.00	1062 260	2.09	820 171	1.59	629 614
	7	3.00	1075 820	2.09	830 637	1.59	634 416
5	8	3.00	1079 830	2.09	833 730	1.59	635 657
	9	3.00	1080 880	2.09	834 543	1.59	635 943
	10	3.00	1081 130	2.09	834 736	1.59	636 002

and 6 that as the failure number threshold for NCCs increases, a shorter warranty length is preferred to minimize the ECR, which is the same trend as shown in tables 3 and 4.

In table 6, we numerically analyze the trend changes of W_L^* and its ECR for several levels of PM effect, denoted by δ. It is noted that the greater the value of δ is, the more effective the PM action is expected to be at each PM action. Table 6 shows that W_L^* increases when l_c increases and l_n decreases, which is the same trend as given in table 5. As for the effects of δ, we observe that the effect of δ on W_L^* and the ECR seems insignificant for fixed values of l_c and l_n, although the ECR

Table 6. Optimal warranty length W_L^* and its corresponding ECR for various combinations of l_c, l_n, and n_p^r for $c_c = 1{,}000, c_n = 400$, $c_{pm} = 500, W_0 = 1.0,$ and $\tau_d = 0.2$.

		$\delta = 0.2$		$\delta = 0.6$		$\delta = 0.9$	
l_c	l_n	W_L^*	ECR	W_L^*	ECR	W_L^*	ECR
	3	1.62	192 560	1.62	197 085	1.62	201 444
	4	1.60	245 480	1.60	249 203	1.59	252 603
2.0	5	1.59	270 010	1.58	272 758	1.58	275 122
	6	1.58	279 396	1.58	281 544	1.57	283 301
	7	1.58	282 472	1.57	284 351	1.57	285 843
	4	1.83	481 174	1.83	482 638	1.82	483 163
	5	1.82	531 263	1.82	530 153	1.81	528 013
3.0	6	1.81	550 723	1.81	548 140	1.81	544 540
	7	1.81	557 184	1.81	553 960	1.80	549 742
	8	1.81	559 063	1.81	555 608	1.80	551 175
	5	2.00	704 004	1.99	695 697	1.98	686 013
	6	1.99	730 780	1.98	720 186	1.97	708 268
4.0	7	1.99	739 756	1.98	728 180	1.97	715 335
	8	1.99	742 388	1.98	730 462	1.97	717 296
	9	1.99	743 075	1.98	731 042	1.97	717 781

decreases slightly for larger values of l_c and l_n. By analyzing the numerical results given in tables 5 and 6, both W_L^* and its ECR are affected more by the inter-PM interval τ_d than by the level of the PM effect δ.

7. Concluding remarks

This paper considers a two-dimensional warranty model for a repairable multi-component system, which is composed of two types of components: critical and non-critical. Under the proposed warranty model, we consider both the warranty region and the R/R region, where the R/R region is set within the warranty region. Moreover, a fixed number of periodic PM actions are performed during the age-based R/R period under the warranty model. All component failures are minimally repaired free to the user during the age-based warranty period, and when the component failures reach certain thresholds (which is referred to as an R/R condition in this paper) within the age-based R/R period, the system is either refunded or replaced, and in the case of replacement, the warranty is renewed. Otherwise, the system is warranted until the expiration of the original warranty. Thus, the life cycle of the system ends either at the time when the R/R condition occurs or at the expiration of the age-based warranty period.

The main objective of this work is to propose an optimal warranty model by searching for an optimal length of the warranty period, during which the ECR is minimized from the manufacturer's perspective. Although extensive research has been done in the field of system maintenance and warranty policy for many years, our proposed warranty model is formulated uniquely by defining the R/R region within the warranty region based on age and usage so that either a refund or replacement of the system can occur subject to satisfaction of certain conditions regarding the component failure

numbers. Such conditions are closely related to the lemon law, which is being enforced in many countries to protect users from frequent system failures. The meaningful contribution of this paper lies in (1) modeling of a two-dimensional warranty scheme based on both age and usage, (2) adaptation of the R/R condition for the R/R of a system, and (3) consideration of periodic PM during the R/R period.

For the sensitivity analysis under the proposed warranty model, we numerically study the effects of relevant parameters, such as the failure number thresholds of CCs and NCCs, minimal repair cost, inter-PM interval, etc, on the optimal warranty length in case a refund occurs within the R/R period. The numerical results show that the optimal warranty length becomes longer as the failure number threshold of CCs increases. This implies that a longer warranty length is more beneficial to the manufacturer in terms of the ECR when the CC threshold is set higher. However, it is noted from tables 3 and 4 that the NCC threshold number works adversely on the determination of the optimal warranty length. Regarding the impact of periodic PM actions, it seems that the inter-PM interval affects the optimal warranty length more significantly than the level of PM effect applied at each PM action. However, such a numerical analysis is based on a limited set of parameter values only, so more comprehensive numerical calculations might be needed to analyze the impact of relevant parameters on the proposed warranty model.

From working on this article for direct application of lemon law warranty policy, we realize that there are still several unsolved theoretical problems for more accurate cost analysis in the field of warranty policy, such as the probability distribution of the elapsed time to the occurrence of refund or replacement. When the PM actions are implemented during the warranty period, the intensity rate function is adjusted after each PM action, and thus the distribution for the number of failures in an interval of $[0, t]$ becomes almost intractable. Research for approximation methods of such a problem would be a great contribution in this field.

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